PHASE MODEL EXPECTATION VALUES AND THE 2-TODA HIERARCHY

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ABSTRACT. We show that the scalar product of the phase model on a finite rectangular lattice is a (restricted) τ-function of the 2-Toda hierarchy. Using this equivalence we then show that the wave-functions of the hierarchy correspond to certain classes of boundary correlation functions of the model.

0. INTRODUCTION

In [1], it was observed that the $N \times N$ domain wall partition function, Z_N, of the six vertex model is, up to a multiplicative factor, a τ-function of the KP hierarchy. In [2], the length M XXZ spin-$\frac{1}{2}$ chain and its associated scalar product, $\langle\{\lambda\}|\{\mu\}\rangle$, were considered. Restricting either the initial or the final state to a Bethe eigenstate, the resulting expression is again a KP τ-function.

In this work we further extend the known correspondences between integrable quantum lattice models and classical hierarchies of non linear partial differential equations in the following way[1]. We show that the scalar product of the phase model [4,5], up to a multiplicative factor, is a (restricted) τ-function of the 2-Toda hierarchy [6,7], where the Toda time variables are power sums of the rapidities. We then consider the two types of wave-functions from the Toda theory, $\hat{w}^{(\infty)}$ and $\hat{w}^{(0)}$, and show that they correspond to a certain class of boundary correlation function from the phase model perspective. We additionally give a single determinant form for each of these correlation functions.

In section **1**, we recall known results about the finite 2-Toda hierarchy, its construction from the wave-matrix *initial value problem* and the τ-function as a finite bilinear sum of character/Schur polynomials. In **2**, we introduce the phase model and the aforementioned lattice model/hierarchy correspondence. Additionally we use known combinatorial bijections to express the state vectors of the model as weighted sums of various objects. In **3**, we use the weighted sum expressions of the state vectors to show that the Toda wave-functions correspond to specific classes of boundary correlation functions and give their single determinant form. In **4**, we offer some remarks.

1. THE FINITE 2-TODA HIERARCHY

1.1. **Definition of the hierarchy.** The details of this section can mostly be found in [6,7]. We begin by giving the definition of the 2-Toda hierarchy, with two sets of $(n-m-1)$ time variables, in terms of four distinct Lax type systems of first order differential equations.

Defining the following shift matrices,

$$\Lambda_{[m,n)}^{\pm j} \equiv (\delta_{k\pm j,l})_{k,l\in\{m,\ldots,n-1\}}$$

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[1]We refer to [3] for an introduction to correspondences between integrable quantum models and differential equations.

the general matrix $A \in gl(n-m)$ is written in the form,

$$A = \sum_{j=-n+m+1}^{n-m-1} a_j(s)\Lambda^j_{[m,n)}$$

where $m \le s \le n-1$ denotes the row of the matrix A. A is said to be a strictly lower triangular, $A = (A)_-$, if $a_j(s) = 0$ for $j \ge 0$ and upper triangular, $A = (A)_+$, if $a_j(s) = 0$ for $j < 0$,

$$(A)_+ = \sum_{j=0}^{n-m-1} a_j(s)\Lambda^j_{[m,n)} \ , \ (A)_- = \sum_{j=-n+m+1}^{-1} a_j(s)\Lambda^j_{[m,n)}$$

We define two sets of time flows $\vec{x}$ and $\vec{y}$ as,

$$\vec{x} = \{x_1, x_2, \dots, x_{n-m-1}\} \ , \ \vec{y} = \{y_1, y_2, \dots, y_{n-m-1}\}$$

and introduce $L(\vec{x},\vec{y}), M(\vec{x},\vec{y}), B_k(\vec{x},\vec{y}), C_k(\vec{x},\vec{y}) \in gl(n-m)$ where,

$$\begin{aligned} L &= \sum_{j=-n+m+1}^{1} b_j(s,\vec{x},\vec{y})\Lambda^j_{[m,n)}, \quad b_1(s) = 1, \qquad B_k = (L^k)_+ \\ M &= \sum_{j=-1}^{n-m-1} c_j(s,\vec{x},\vec{y})\Lambda^j_{[m,n)}, \qquad c_{-1}(s) \neq 0, \quad C_k = (M^k)_- \end{aligned}$$

We define the 2-Toda hierarchy as the following Lax type system of differential equations,

$$\partial_{x_k} L = [B_k, L] \ , \ \partial_{y_k} L = [C_k, L] \ , \ \partial_{x_k} M = [B_k, M] \ , \ \partial_{y_k} M = [C_k, M] \tag{1}$$

or equivalently (theorem 1.1 of [6]), the Zakharov-Shabat system,

$$\begin{aligned} &\partial_{x_j} B_k - \partial_{x_k} B_j + [B_k, B_j] = 0 \ , \ \partial_{y_j} C_k - \partial_{y_k} C_j + [C_k, C_j] = 0 \\ &\partial_{y_j} B_k - \partial_{x_k} C_j + [B_k, C_j] = 0 \end{aligned} \tag{2}$$

Compatibility conditions. It can be shown that the above systems are the compatibility conditions of the linear operator equations,

$$\partial_{x_j} W^{(\infty/0)} = B_j W^{(\infty/0)} \ , \ \partial_{y_j} W^{(\infty/0)} = C_j W^{(\infty/0)} \tag{3}$$

where $W^{(\infty/0)} = W^{(\infty/0)}(\vec{x},\vec{y}) \in GL(n-m)$ are referred to as *wave-matrices.*

1.2. **The initial value problem.** By defining the constant matrix $A \in GL(n-m) = (a_{ij})_{i,j=m,\dots,n-1}$, such that $\det[a_{ij}]_{i,j=m\dots,s-1} \neq 0$, $m < s \le n$, it is possible to find wave-matrices, $W^{(\infty)}$ and $W^{(0)}$, such that

$$W^{(0)} = W^{(\infty)} A \tag{4}$$

where $W^{(\infty)}$ and $W^{(0)}$ have the specific form

$$\begin{aligned} W^{(\infty)} &= \hat{W}^{(\infty)} \exp\left[\sum_{k=1}^{n-m-1} x_k \Lambda^k_{[m,n)}\right] \ , \ \hat{W}^{(\infty)} = \left(\hat{w}^{(\infty)}_{i-j}(i,\vec{x},\vec{y})\right)_{m \le i,j \le n-1} \\ W^{(0)} &= \hat{W}^{(0)} \exp\left[\sum_{k=1}^{n-m-1} y_k (\Lambda^T_{[m,n)})^k\right] \ , \ \hat{W}^{(0)} = \left(\hat{w}^{(0)}_{j-i}(i,\vec{x},\vec{y})\right)_{m \le i,j \le n-1} \end{aligned}$$

$$\hat{w}^{(\infty)}_j = \begin{cases} 0 & j < 0 \\ 1 & j = 0 \end{cases} \ , \ \hat{w}^{(0)}_j = \begin{cases} 0 & j < 0 \\ \hat{w}^{(0)}_j(\vec{x},\vec{y}) \neq \mathit{const.} & j = 0 \end{cases} \tag{5}$$

The remaining non zero entries of the wave-matrices, $\hat{W}^{(\infty)}$ and $\hat{W}^{(0)}$, are given by (proposition 3.1 of [7]),

$$\begin{aligned} \hat{w}^{(\infty)}_k(s,\vec{x},\vec{y}) &= (-1)^k \frac{\det[a_{ij}(\vec{x},\vec{y})]_{\substack{i=m,\dots,\widehat{s-k},\dots,s \\ j=m,\dots,s-1}}}{\det[a_{ij}(\vec{x},\vec{y})]_{i,j=m,\dots,s-1}}, \ \text{for} \begin{cases} 0 \le k \le s-m \\ m < s \le n-1 \end{cases} \\ \hat{w}^{(0)}_k(s,\vec{x},\vec{y}) &= \frac{\det[a_{ij}(\vec{x},\vec{y})]_{\substack{i=m,\dots,s \\ j=m,\dots,s-1,s+k}}}{\det[a_{ij}(\vec{x},\vec{y})]_{i,j=m,\dots,s-1}}, \ \text{for} \begin{cases} 0 \le k \le n-s-1 \\ m < s \le n-1 \end{cases} \end{aligned} \tag{6}$$

where,

$$(a_{ij}(\vec{x},\vec{y}))_{m\le i,j\le n-1} = \exp\left[\sum_{k=1}^{n-m-1} x_k \Lambda_{[m,n)}^k\right] A \exp\left[-\sum_{k=1}^{n-m-1} y_k (\Lambda_{[m,n)}^T)^k\right]$$

and s refers to the row of the entry. The entries of the inverse of the wave-matrices $\left(\hat{W}^{(0)}(\vec{x},\vec{y})\right)^{-1}$ and $\left(\hat{W}^{(\infty)}(\vec{x},\vec{y})\right)^{-1}$ are similarly given as

(7)
$$\hat{w}_k^{*(0)}(s,\vec{x},\vec{y}) = (-1)^k \frac{\det\left[a_{ij}(\vec{x},\vec{y})\right]_{\substack{i=m,\ldots,s-1\\ j=m,\ldots,\hat{s-k},\ldots,s}}}{\det\left[a_{ij}(\vec{x},\vec{y})\right]_{i,j=m,\ldots,s}}, \text{ for } \begin{cases} 0\le k\le s-m \\ m<s\le n-1 \end{cases}$$
$$\hat{w}_k^{*(\infty)}(s,\vec{x},\vec{y}) = \frac{\det\left[a_{ij}(\vec{x},\vec{y})\right]_{\substack{i=m,\ldots,s-1,s+k\\ j=m,\ldots,s}}}{\det\left[a_{ij}(\vec{x},\vec{y})\right]_{i,j=m,\ldots,s}}, \text{ for } \begin{cases} 0\le k\le n-s-1 \\ m<s\le n-1 \end{cases}$$

where s now refers to the column of the entry, as opposed to the row.

The generalized Lax and Zakharov-Shabat systems. From proposition 3.2 in [7], the following matrices,

$$L = W^{(\infty)}\Lambda_{[m,n)}\left(W^{(\infty)}\right)^{-1},\ M = W^{(0)}\Lambda_{[m,n)}^T\left(W^{(0)}\right)^{-1}$$

and $B_k = \left\{L^k\right\}_+$, $C_k = \left\{M^k\right\}_-$, satisfy the linear operator equations (eq. 3), the Zakharov-Shabat equations (eq. 2) and the Lax equations (eq. 1) which define the 2-Toda hierarchy.

1.3. **Tau-function of the 2-Toda hierarchy.** The τ-function, $\tau(s,\vec{x},\vec{y})$, is a function of the time parameters $\vec{x}$ and $\vec{y}$ and an additional parameter, s, which corresponds to the row number of $W^{(\infty)}, W^{(0)}$ or the column number of $\left(W^{(\infty)}\right)^{-1}$, $\left(W^{(0)}\right)^{-1}$. The derivatives of $\tau(s,\vec{x},\vec{y})$ correspond to the entries of the wave-matrices and using this fact, we can express the 2-Toda hierarchy in a single integral bilinear form.

Proposition 1. *For the function,*

(8)
$$\tau(s,\vec{x},\vec{y}) = det[a_{ij}(\vec{x},\vec{y})]_{m\le i,j\le s-1}$$

the following four relations hold,

(9)
$$\hat{w}_k^{(\infty)}(s) = \frac{\zeta_k\left(-\tilde{\partial}_{\vec{x}}\right)\tau(s,\vec{x},\vec{y})}{\tau(s,\vec{x},\vec{y})} \qquad \hat{w}_k^{(0)}(s) = \frac{\zeta_k\left(-\tilde{\partial}_{\vec{y}}\right)\tau(s+1,\vec{x},\vec{y})}{\tau(s,\vec{x},\vec{y})}$$
$$\hat{w}_k^{*(\infty)}(s) = \frac{\zeta_k\left(\tilde{\partial}_{\vec{x}}\right)\tau(s+1,\vec{x},\vec{y})}{\tau(s+1,\vec{x},\vec{y})} \qquad \hat{w}_k^{*(0)}(s) = \frac{\zeta_k\left(\tilde{\partial}_{\vec{y}}\right)\tau(s,\vec{x},\vec{y})}{\tau(s+1,\vec{x},\vec{y})}$$

where,

$$\tilde{\partial}_{\vec{x}} = \left(\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \ldots\right),\ \tilde{\partial}_{\vec{y}} = \left(\partial_{y_1}, \frac{1}{2}\partial_{y_2}, \frac{1}{3}\partial_{y_3}, \ldots\right)$$

and the generating function for the ***one row character polynomial****, $\zeta_k(\vec{x})$, is given by,*

(10)
$$\sum_{k=0}^{\infty} z^k \zeta_k(\vec{x}) = \exp\left\{\sum_{j=1}^{n-m-1} z^j x_j\right\}$$

Proof. If the above four relations are true then their weighted summations are given by,

(11)
$$\sum_{k=0}^{s-m} \lambda^k \hat{w}_k^{(\infty)}(s) = \frac{\tau(s,\vec{x}-\vec{\epsilon}(\lambda),\vec{y})}{\tau(s,\vec{x},\vec{y})} \qquad \sum_{k=0}^{n-s-1} \lambda^k \hat{w}_k^{(0)}(s) = \frac{\tau(s+1,\vec{x},\vec{y}-\vec{\epsilon}(\lambda))}{\tau(s,\vec{x},\vec{y})}$$
$$\sum_{k=0}^{n-s-1} \lambda^k \hat{w}_k^{*(\infty)}(s) = \frac{\tau(s+1,\vec{x}+\vec{\epsilon}(\lambda),\vec{y})}{\tau(s+1,\vec{x},\vec{y})} \qquad \sum_{k=0}^{s-m} \lambda^k \hat{w}_k^{*(0)}(s) = \frac{\tau(s,\vec{x},\vec{y}+\vec{\epsilon}(\lambda))}{\tau(s+1,\vec{x},\vec{y})}$$

where, $\vec{\epsilon}(\lambda) = (\lambda, \frac{\lambda^2}{2}, \frac{\lambda^3}{3}, \dots)$. By using the methods in proposition 3.4 of [7] we explicitly obtain,

$$\tau(s, \vec{x} \mp \vec{\epsilon}(\lambda), \vec{y}) = \det\left[(1 - \lambda\Lambda_{[m,n)})^{\pm 1} A(\vec{x}, \vec{y})\right]_{m \le i,j \le s-1}$$
$$\tau(s, \vec{x}, \vec{y} \mp \vec{\epsilon}(\lambda)) = \det\left[A(\vec{x}, \vec{y})(1 - \lambda\Lambda^T_{[m,n)})^{\mp 1}\right]_{m \le i,j \le s-1}$$

which upon expanding as a polynomial in λ we obtain the required summations in eq. 11. □

2-Toda Bilinear relation. The function $\tau(s, \vec{x}, \vec{y})$ defined in eq. 8 satisfies the following bilinear relation,

(12)
$$\oint \frac{d\lambda}{2\pi i} \lambda^{s'-s-2} \exp\left\{\sum_{l=1}^{n-m-1} (y_l - y'_l)\lambda^l\right\} \frac{\tau\left(s+1, \vec{x}, \vec{y} - \vec{\epsilon}\left(\frac{1}{\lambda}\right)\right)}{\tau(s, \vec{x}, \vec{y})} \frac{\tau\left(s'-1, \vec{x}', \vec{y}' + \vec{\epsilon}\left(\frac{1}{\lambda}\right)\right)}{\tau\left(s', \vec{x}', \vec{y}'\right)}$$
$$= \oint \frac{d\lambda}{2\pi i} \lambda^{s-s'} \exp\left\{\sum_{l=1}^{n-m-1} (x_l - x'_l)\lambda^l\right\} \frac{\tau\left(s, \vec{x} - \vec{\epsilon}\left(\frac{1}{\lambda}\right), \vec{y}\right)}{\tau\left(s, \vec{x}, \vec{y}\right)} \frac{\tau\left(s', \vec{x}' + \vec{\epsilon}\left(\frac{1}{\lambda}\right), \vec{y}'\right)}{\tau\left(s', \vec{x}', \vec{y}'\right)}$$

for general $s, s', \vec{x}, \vec{x}', \vec{y}, \vec{y}'$. The integration $\oint \frac{d\lambda}{2\pi i}$ simply refers to the algebraic operation of obtaining the coefficient of $\frac{1}{\lambda}$.

Polynomial expressions of the τ-function. Rewriting the shift matrix exponentials appropriately,

$$\exp\left\{\sum_{l=1}^{n-m-1} x_l \Lambda^l_{[m,n)}\right\} = \sum_{j=0}^{n-m-1} \zeta_j(\vec{x}) \Lambda^j_{[m,n)} = \left(\zeta_{j-i}(\vec{x})\right)_{m \le i,j \le n-1}$$
$$\exp\left\{-\sum_{l=1}^{n-m-1} y_l \left(\Lambda^T_{[m,n)}\right)^l\right\} = \sum_{j=0}^{n-m-1} \zeta_j(-\vec{y}) \left(\Lambda^T_{[m,n)}\right)^j = \left(\zeta_{i-j}(-\vec{y})\right)_{m \le i,j \le n-1}$$

and using the repeated application of the Cauchy-Binet identity, the τ-function becomes,

$$\tau\left(s, \vec{x}, \vec{y}\right) = \sum_{\{\lambda\}\{\mu\} \subseteq (n-s)^{(s-m)}} A_{\{\lambda\}\{\mu\}} \chi_{\{\lambda\}}(\vec{x}) \chi_{\{\mu\}}(-\vec{y}) \tag{13}$$

where $\{\lambda\}$ and $\{\mu\}$ are partitions contained within the box of dimensions $(n-s)^{(s-m)}$, $\chi_{\{\lambda\}}(\vec{x})$ is the character polynomial given by,

$$\chi_{\{\lambda\}}(\vec{x}) = \det\left[\zeta_{\lambda_i + j - i}(\vec{x})\right]_{1 \le i,j \le s-m} \tag{14}$$

and,

$$A_{\{\lambda\}\{\mu\}} = \det\left[a_{\lambda_{s-m+1-i}+i+m-1, \mu_{s-m+1-j}+j+m-1}\right]_{1 \le i,j \le s-m}$$

1.4. **Restricting the time variables.** In order to make contact with the phase model, it is necessary to restrict the time variables in such a way that the τ-function becomes an element of the symmetric polynomial ring,
$\mathbb{C}\{[u_1, \dots, u_{s-m}]^{S_{s-m}}, [v_1, \dots, v_{s-m}]^{S_{s-m}}\}$.

In the remainder of this work we shall use the convention that,

- $\tau(\vec{x}, \vec{y})$ denotes that the time variables are algebraically independent.
- $\tau(\vec{u}, \vec{v})$ denotes that the time variables are algebraically dependent, and $\tau(\vec{u}, \vec{v})$ is an element of the aforementioned symmetric polynomial ring. We shall refer to $\tau(\vec{u}, \vec{v})$ as a **restricted** τ-function.

Schur polynomials. Changing from time parameters to symmetric power sums[2],

$$x_k = \frac{1}{k} p_k(u_1, \dots, u_{s-m}) \ , \ -y_k = \frac{1}{k} p_k(v_1, \dots, v_{s-m}) \ , \ k \in \{1, \dots, n-m-1\} \tag{15}$$

[2]We define the symmetric power sum of order k as, $p_k(\vec{u}) = \sum_{j=1}^{s-m} u_j^k$.

the one row character polynomials, $\zeta_i(\vec{x})$ and $\zeta_i(-\vec{y})$, become complete homogeneous symmetric polynomials[3],

$$\zeta_i(\vec{x}) \to h_i(u_1,\dots,u_{s-m}) \quad , \quad \zeta_i(-\vec{y}) \to h_i(v_1,\dots,v_{s-m})$$

Hence the character polynomials in the τ-function become Schur polynomials,

$$\tau\left(s,\vec{u},\vec{v}\right) = \sum_{\{\lambda\}\{\mu\}\subseteq(n-s)^{(s-m)}} A_{\{\lambda\}\{\mu\}} S_{\{\lambda\}}(\vec{u}) S_{\{\mu\}}(\vec{v}) \tag{16}$$

2. The phase model

Most of the material in this section can be found in [4, 5]. Consider the bosonic algebra[4] generated by the three operators ϕ, $\phi^\dagger$ and N that satisfy the following commutation relations,

$$[\phi,\phi^\dagger] = \pi \ , \ [N,\phi] = -\phi \ , \ [N,\phi^\dagger] = \phi^\dagger$$

where $\pi = |0\rangle\langle 0|$ is the vacuum projector. The one dimensional Fock space, $\mathbb{F}$, for this algebra is formed from the state $|n\rangle$, where the label $n \in \mathbb{Z}_+ \cup \{0\}$ is called an occupation number. The action of the operators on elements of the Fock space are given by,

$$\phi^\dagger|n\rangle = |n+1\rangle \ , \ \phi|n\rangle = |n-1\rangle \ , \ N|n\rangle = n|n\rangle$$

The action of the ϕ operator on the vacuum state, $|0\rangle$, annihilates it.

We now extend the above bosonic algebra and consider the tensor product,

$$\mathbb{F} = \mathbb{F}_0 \otimes \mathbb{F}_1 \otimes \cdots \otimes \mathbb{F}_M$$

We introduce the operators, ϕ_j, $\phi_j^\dagger$ and N_j, $0 \le j \le M$, that act on $\mathbb{F}_j$ as,

$$\phi_j = I_0 \otimes I_1 \otimes \dots I_{j-1} \otimes \phi \otimes I_{j+1} \otimes \cdots \otimes I_M$$

and similarly for $\phi_j^\dagger$ and N_j, where I_k is the identity operator in $\mathbb{F}_k$. The commutation relations are given by

$$[\phi_j,\phi_k^\dagger] = \pi_j\delta_{jk} \ , \ [N_j,\phi_k] = -\phi_j\delta_{jk} \ , \ [N_j,\phi_k^\dagger] = \phi_j^\dagger\delta_{jk} \tag{17}$$

where each operator of index j acts on the corresponding indexed Fock vectors,

$$\begin{array}{lcl} (\phi_j)^{m_j-n_j}|m_j\rangle_j & = & |n_j\rangle_j \text{ for } 0 \le n_j < m_j \\ \left(\phi_j^\dagger\right)^{n_j-m_j}|m_j\rangle_j & = & |n_j\rangle_j \text{ for } n_j > m_j \ge 0 \\ N_j|m_j\rangle_j & = & m_j|m_j\rangle_j \end{array} \tag{18}$$

and ϕ_j annihilates the vacuum state $|0\rangle_j$. The state vectors, $|n_p\rangle_j$, and the corresponding conjugate vectors, $\langle n_r|_k$, are orthonormal,

$$\langle n_r|n_p\rangle_{k,j} = \delta_{pr}\delta_{jk} \tag{19}$$

2.1. **Algebraic Bethe ansatz.**

We define the phase model through the following local L-operator matrix,

$$L_j(u) \equiv \begin{pmatrix} \hat{a}_j(u) & \hat{b}_j(u) \\ \hat{c}_j(u) & \hat{d}_j(u) \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & \phi_j^\dagger \\ \phi_j & u \end{pmatrix} \tag{20}$$

where $u \in \mathbb{C}$. Naturally associated with $L_j(u)$ is the 4×4 R-matrix,

$$R(u,v) = \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & g(u,v) & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix} \tag{21}$$

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[3]The complete homogeneous symmetric polynomials, $h_k(\vec{u})$, are generated by $\sum_{k=0}^{\infty} z^k h_k(\vec{u}) = \prod_{j=1}^{s-m} \frac{1}{1-zu_j} = \exp\left\{\sum_{j=1}^{\infty} z^j \frac{1}{j} p_j(\vec{u})\right\}$. For additional information see section I.2 of [8].

[4]We note that this algebra is the $q=0$ limit of the q-boson algebra.

where $f(u,v) = \frac{u^2}{u^2-v^2}$ and $g(u,v) = \frac{uv}{u^2-v^2}$. L and R satisfy the following intertwining relation,

$$R(u,v)[L_j(u) \otimes L_j(v)] = [L_j(v) \otimes L_j(u)]R(u,v) \tag{22}$$

The monodromy matrix. The monodromy matrix, $T(u)$, for the phase model is introduced as the ordered product of all $(M+1)$ L-matrices,

$$T(u) = L_M(u)L_{M-1}(u)\dots L_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{23}$$

Using induction on the intertwining relation (eq. 22) the monodromy matrix and the R-matrix satisfy an equivalent intertwining relationship,

$$R(u,v)[T(u) \otimes T(v)] = [T(v) \otimes T(u)]R(u,v) \tag{24}$$

which generate sixteen non trivial algebraic relationships.

Non local creation and annihilation operators. Focusing on the operators $B(u)$ and $C(u)$, we consider the operator $\hat{N} = \sum_{j=0}^{M} N_j$, which measures the total occupation number of the state. Applying this operator to $B(u)$ and $C(u)$ we obtain,

$$\hat{N}B(u) = B(u)\left\{\hat{N}+1\right\} \ , \hat{N}C(u) = C(u)\left\{\hat{N}-1\right\} \tag{25}$$

Thus the operator $B(u)$ is a creation operator of the phase model, where one application on a state vector increases the total occupation number by one, while $C(u)$ is the opposing annihilation operator of the phase model, where one application to a state vector decreases the total occupation number by one. We note that $C(u)$ annihilates the total vacuum operator.

Equivalently, the roles of $B(u)$ and $C(u)$ are reversed when applied to the conjugated vacuum vectors.

2.2. **N-particle state vector and its conjugate.** We construct the N-particle vector, $|\Psi_M\rangle$, by repeated application of the construction operator B on the vacuum vector,

$$|\Psi_M\rangle = B(u_1)\dots B(u_N)|0\rangle$$

where the total occupation number of $|\Psi_M\rangle$ is N. An alternative form for the N-particle vector is given by,

$$|\Psi_M\rangle = \sum_{\substack{0\le n_0,n_1,\dots,n_M\le N \\ n_0+n_1+\dots+n_M=N}} f_{\{n_0,\dots,n_M\}}(\vec{u}) \prod_{k=0}^{M} \left(\phi_k^\dagger\right)^{n_k} \bigotimes_{j=0}^{M} |0\rangle_j = \sum_{\{\lambda\}\subseteq(M)^N} f_{\{\lambda\}}(\vec{u})|\lambda\rangle \tag{26}$$

where the partition $\{\lambda\}$ is constructed from the occupation number sequence in the following manner,

$$\{\lambda\} = (M^{n_M}, (M-1)^{n_{M-1}}, \dots, 1^{n_1}, 0^{n_0}) \tag{27}$$

Similarly, we construct the conjugate N-particle vector, $\langle\Psi_M|$, by repeated application of the annihilation operator C on the conjugate vacuum vector,

$$\langle\Psi_M| = \langle 0|C(v_N)\dots C(v_1)$$

where the total occupation number N, and an alternative form is given by,

$$\langle\Psi_M| = \sum_{\substack{0\le n_0,n_1,\dots,n_M\le N \\ n_0+n_1+\dots+n_M=N}} g_{\{n_0,\dots,n_M\}}(\vec{v}) \bigotimes_{j=0}^{M} \langle 0|_j \prod_{k=0}^{M} (\phi_k)^{n_k} = \sum_{\{\lambda\}\subseteq(M)^N} g_{\{\lambda\}}(\vec{v})\langle\lambda| \tag{28}$$

Schur polynomial expansion of state vectors. In [9], Tsilevich derived the following Schur polynomial forms for $f_{\{\lambda\}}(\vec{u})$ and $g_{\{\lambda\}}(\vec{v})$ in eqs. 26 and 28,
(29)
$$f_{\{\lambda\}}(\vec{u}) = \left(\prod_{j=1}^{N} u_j\right)^{-M} S_{\{\lambda\}}(u_1^2, \ldots, u_N^2) \ , \ g_{\{\lambda\}}(\vec{v}) = \left(\prod_{j=1}^{N} v_j\right)^{M} S_{\{\lambda\}}(v_1^{-2}, \ldots, v_N^{-2})$$
In order to proceed, we need to express the state vectors as weighted sums of lattice configurations, plane partitions and semi-standard tableaux.

2.3. **Weighted sums of lattice paths.** Consider N non-crossing column strict lattice paths on the $(M+1) \times 2N$ lattice, where the paths begin at the first N bottom-most horizontal edges, $(-N, 0), (-N+1, 0), \ldots, (-1, 0)$, and end at the final N top-most horizontal edges, $(1, M), (2, M), \ldots, (N, M)$, respectively. Each path is restricted to move either up or right, and no two paths can cross or occupy the same vertical edge. We give a typical example in fig. 1.

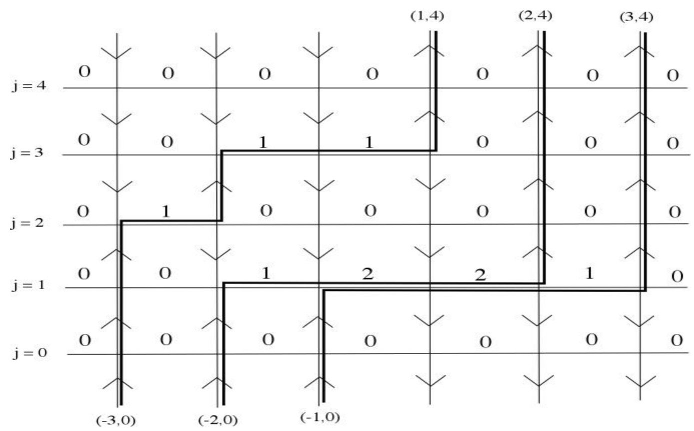

FIGURE 1. Generic lattice path configuration for $M = 4$, $N = 3$.

Given an allowable lattice path configuration, we assign each of the four possible vertices a letter, as indicated in fig. 2.

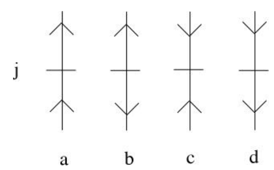

FIGURE 2. The four vertices used to construct lattice paths.

Following section V of [5], it is possible to express the coefficient of the conjugate state vector, $g_{\{n_0,\ldots,n_M\}}(\vec{v})$, as the following weighted sum of lattice paths,

$$g_{\{n_{j_1},\ldots,n_{j_k}\}}(\vec{v}) = \sum_{\substack{\text{allowable paths} \\ \text{in } (M+1)\times N \text{ lattice}}} v_1^{t_1^d - t_1^a} v_2^{t_2^d - t_2^a} \ldots v_N^{t_N^d - t_N^a} \tag{30}$$

where the sum is taken over all allowable paths in the $(M+1) \times N$ lattice under the conditions,

- n_{j_1} paths start at $(-N, 0), (-N-1, 0), \ldots, (-N+n_{j_1}-1, 0)$ and end at $(-1, j_1)$.

- n_{j_2} paths start at $(-N+n_{j_1},0),\ldots,(-N+\sum_{l=1}^{2} n_{j_l}-1,0)$ and end at $(-1,j_2)$.
- This procedure continues until we finally have n_{j_k} paths starting at $(-N+\sum_{l=1}^{k-1} n_{j_l},0),\ldots,(-1,0)$ and ending at $(-1,j_k)$.

The powers t_l^d and t_l^a, $1 \le l \le N$, are equal to the number of d and a vertices respectively in the lth column.

Similarly, the coefficient of the state vector, $f_{\{n_0,\ldots,n_M\}}(\vec{u})$, can be expressed as the following weighted sum of lattice paths,

$$f_{\{n_{j_1},\ldots,n_{j_k}\}}(\vec{u}) = \sum_{\substack{\text{allowable paths}\\ \text{in } (M+1)\times N \text{ lattice}}} u_1^{t_1^d-t_1^a} u_2^{t_2^d-t_2^a} \ldots u_N^{t_N^d-t_N^a} \tag{31}$$

where the sum is taken over all allowable paths in the $(M+1)\times N$ lattice under the conditions,

- n_{j_1} paths start at $(1,j_1)$ and end at $(1,M),(2,M),\ldots,(n_{j_1},M)$.
- n_{j_2} paths start at $(1,j_2)$ and end at $(n_{j_1}+1,M),\ldots,(\sum_{l=1}^{2} n_{j_l},M)$.
- This procedure continues until we finally have n_{j_k} paths starting at $(1,j_k)$ and ending at $(\sum_{l=1}^{k-1} n_{j_l}+1,M),\ldots,(N,M)$.

2.4. **Weighted sums of plane partitions.** A plane partition, $\pi_{j,k}$, is an array of non negative integers such that,

$$\pi_{j,k} \ge \pi_{j+1,k} \text{ and } \pi_{j,k} \ge \pi_{j,k+1}$$

If we restrict the size of the array to be $N\times N$, and restrict the maximum of any integer within the plane partition, $\pi_{i,j} \le M$, the plane partition is said to be contained within a box of side lengths $N\times N\times M$.

A typical example of a plane partition within a box of $3\times 3\times 4$ is given by the following[5],

$$\pi^{\{\lambda'\}} = \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \tag{32}$$

Note that the diagonal entries of any plane partition always give a partition in usual sense[6]. The graphical representation of a plane partition in a $N\times N\times M$ box is given by considering rhombus tilings of a (N,N,M) semiregular hexagon. The plane partition, $\pi^{\{\lambda'\}}$, is represented by fig. 3, where each representation can be generally constructed entirely from the three types of rhombi given in fig. 4.

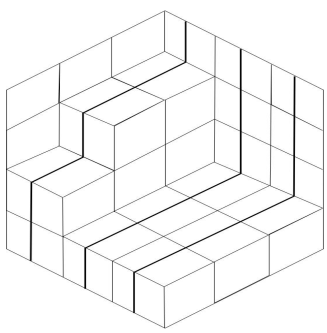

FIGURE 3. Graphical representation of the plane partition $\pi^{\{\lambda'\}}$.

[5]We shall use the following plane partition, $\pi^{\{\lambda'\}}$, as a running example in this section.

[6]In the above example we obviously have $\{\lambda'\} = (3,1,1)$.

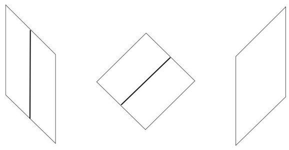

FIGURE 4. The three types of rhombi used to construct plane partitions.

Correspondence between plane partitions in a $\mathbf{N \times N \times M}$ box and N non-crossing column strict lattice paths on the $(\mathbf{M+1}) \times \mathbf{2N}$ lattice. The jth path of the lattice configuration can be thought of as the jth column of the array π^{λ}. As an example, consider the array, $\pi^{\{\lambda'\}}$, which is in correspondence with the lattice path configuration shown in fig. 1.

The bottom left entry, $\pi_{3,1}^{\{\lambda'\}} = 2$, corresponds to the first horizontal section of the first path on the second row, and the remaining entries of the column, $(\pi_{2,1}^{\{\lambda'\}} = 3, \pi_{1,1}^{\{\lambda'\}} = 3)$, correspond to the remaining horizontal sections of the first path, on the third row.

There exists a similar correspondence between the second and third lattice paths, and the second and third columns of the array $\pi^{\{\lambda'\}}$ respectively. It should be clear how this process is generalized for any N and M.

Due to the above correspondence, when considering *lower diagonal plane partitions in an $N \times N \times M$ box*, (equivalently, the *left hand side of the* (N, N, M) *rhombus tiling*), we obtain

$$g_{\{n_{j_1},\ldots,n_{j_k}\}}(\vec{v}) = \sum_{\substack{\text{lower diagonal}\\ \text{plane partitions}}} v_1^{l_1^d - l_1^a} v_2^{l_2^d - l_2^a} \ldots v_N^{l_N^d - l_N^a} \tag{33}$$

where the sum is taken over all allowable lower diagonal $N \times N \times M$ plane partitions, (left hand side (N, N, M) rhombus tilings), and the diagonal terms are given by the partition representation of the corresponding occupation number sequence (eq. 27). The powers l_l^d and l_l^a, $1 \leq l \leq N$, are equal to the number of d and a rhombi respectively in the lth column of the left half rhombus tiling.

Similarly, when considering *upper diagonal plane partitions in an $N \times N \times M$ box*, (equivalently, the *right hand side of the* (N, N, M) *rhombus tiling*), we obtain

$$f_{\{n_{j_1},\ldots,n_{j_k}\}}(\vec{u}) = \sum_{\substack{\text{upper diagonal}\\ \text{plane partitions}}} u_1^{l_1^d - l_1^a} u_2^{l_2^d - l_2^a} \ldots u_N^{l_N^d - l_N^a} \tag{34}$$

where the sum is taken over all allowable upper diagonal $N \times N \times M$ plane partitions. Again, the diagonal terms are given by the partition representation of the corresponding occupation number sequence.

2.5. **Weighted sums of semi-standard tableaux.** We now give the final alternative forms of the state vectors using the following correspondences.

Correspondence between upper diagonal plane partitions, $\pi_{+}^{\{\lambda\}}$, and semi-standard tableaux of descending order[7], $\mathbf{T}_{-}^{\{\lambda\}}$. We begin by considering a general upper half plane partition, $\pi_{+}^{\{\lambda\}}$, and construct a partition using the diagonal entries,

$$\{\lambda\} = \{\pi_{1,1}, \pi_{2,2}, \ldots, \pi_{N,N}\}$$

Considering the next upper diagonal entries, $\pi_{j,j+1}$, we construct the skew diagram, $\{\mu_1\}$,

$$\{\mu_1\} = \{\pi_{1,1} - \pi_{1,2}, \pi_{2,2} - \pi_{2,3}, \ldots, \pi_{N-1,N-1} - \pi_{N-1,N}, \pi_{N,N}\}$$

[7]Semi-standard tableaux are commonly of ascending numerical order, however, descending numerical order is the most convenient convention for the purposes of the next section.

and place the integer 1 in the valid regions of the skew diagram[8]. We then consider the next upper diagonal entries of the array, $\pi_{j,j+2}$, and construct the skew diagram, $\{\mu_2\}$,

$$\{\mu_2\} = \{\pi_{1,1} - \pi_{1,3}, \pi_{2,2} - \pi_{2,4}, \ldots, \pi_{N-2,N-2} - \pi_{N,N-2}, \pi_{N-1,N-1}, \pi_{N,N}\}$$

and place the integer 2 in the valid regions of the skew diagram that have not already been occupied by previous steps in this process. This process continues until the partition contains the numbers $\{1, \ldots, N-1\}$. We then fill the remaining boxes in the partition with the integer N, thereby constructing a valid descending semi-standard tableau $T_-^{\{\lambda\}}$ from the upper diagonal plane partition $\pi_+^{\{\lambda\}}$.

As an example, consider the array, $\pi^{\lambda'}$, given in the past examples where $\{\lambda'\} = (3,1,1)$. The construction of the corresponding descending semi-standard tableau is given in fig. 5.

In **(a)** we construct the partition $\{\lambda'\} = (3,1,1)$. In **(b)** we construct the skew partition $\{\mu_1\} = (3,1,1) - (1,1,0)$ and place the integer 1 in the valid regions of $\{\mu_1\}$. The partition $(1,1,0)$ was obtained from the first upper diagonal entries of $\pi^{\lambda'}$. In **(c)** we construct the skew partition $\{\mu_2\} = (3,1,1) - (1,0,0)$ and place the integer 2 in the valid regions of $\{\mu_2\}$ that contain no integers. The partition $(1,0,0)$ was obtained from the second upper diagonal entries of $\pi^{\lambda'}$. In **(d)** we place the integer 3 in any remaining entries of $\{\lambda'\}$ that don't already contain integers, forming the valid descending semi-standard tableau $T_-^{\{\lambda'\}}$ from the upper diagonal plane partition $\pi_+^{\{\lambda'\}}$.

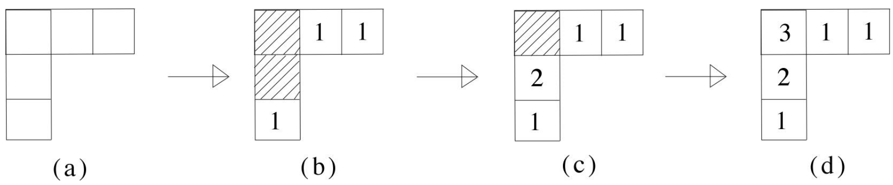

FIGURE 5. Construction of the tableau $T_-^{\{\lambda'\}}$.

Thus based on the above correspondence, another valid combinatorial definition for $f_{\{\lambda\}}(\vec{u})$ is the following,

$$f_{\{\lambda\}}(\vec{u}) = \sum_{T_-^{\{\lambda\}}} u_1^{2t_1 - M} u_2^{2t_2 - M} \ldots u_N^{2t_N - M} \tag{35}$$

where the summation is over all semi-standard Young tableaux of shape $\{\lambda\}$. The powers, t_j, give the weights of $T_-^{\{\lambda\}}$, which count the number of times j appears in the tableau. Note that this expression is in accordance with eq. 29.

Correspondence between lower diagonal plane partitions, $\pi_-^{\{\lambda\}}$, and semi-standard tableaux of ascending order, $\mathbf{T}_+^{\{\lambda\}}$. Using an equivalent algorithm as described above, except this time applying a numerically ascending convention, we obtain the required correspondence. Using $\pi^{\{\lambda'\}}$ as an example yet again, the construction of the corresponding ascending semi-standard tableau is given in fig. 6.

In **(a)** we construct the partition $\{\lambda'\} = (3,1,1)$. In **(b)** we construct the skew partition $\{\nu_1\} = (3,1,1) - (3,1,0)$ and place the integer 3 in the valid regions of $\{\nu_1\}$. The partition $(3,1,0)$ was obtained from the first lower diagonal entries of $\pi^{\lambda'}$. In **(c)** we construct the skew partition $\{\nu_2\} = (3,1,1) - (2,0,0)$ and place

[8]In ascending tableaux, N would be placed instead of 1.

the integer 2 in the valid regions of $\{\nu_2\}$ that contain no integers. The partition $(2,0,0)$ was obtained from the second lower diagonal entries of $\pi^{\lambda'}$. In **(d)** we place the integer 1 in any remaining entries of $\{\lambda'\}$ that don't already contain integers, forming the valid ascending semi-standard tableau $T_+^{\{\lambda'\}}$ from the lower diagonal plane partition $\pi_-^{\{\lambda'\}}$.

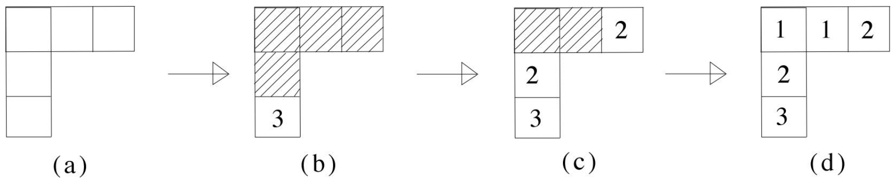

FIGURE 6. Construction of the tableau $T_+^{\{\lambda'\}}$.

Thus we immediately obtain,

$$g_{\{\lambda\}}(\vec{v}) = \sum_{T_+^{\{\lambda\}}} v_1^{-2t_1+M} v_2^{-2t_2+M} \dots v_N^{-2t_N+M} \tag{36}$$

where the summation is over all semi-standard Young tableaux of shape $\{\lambda\}$ of ascending numerical order.

2.6. **The scalar product.** We now consider the scalar product, $\mathbb{S}(N,M|\vec{u},\vec{v})$, of the phase model which is defined as the inner product of the state vectors,

$$\mathbb{S}(N,M|\vec{u},\vec{v}) = \langle \Psi_M | \Psi_M \rangle \tag{37}$$

Using the algebraic expressions from eq. 24, it is possible to obtain the following determinant expression,

$$\mathbb{S}(N,M) = \left(\prod_{i=1}^{N} \frac{1}{u_i v_i}\right)^M \prod_{1\le j<k\le N} \left(\frac{1}{u_j^2 - u_k^2}\right)\left(\frac{1}{v_j^2 - v_k^2}\right) \det\left[h_{M+N-1}(u_m^2, v_l^2)\right]_{1\le m,l\le N} \tag{38}$$

Alternatively, considering the Schur polynomial expansion of the state vectors,

$$\mathbb{S}(N,M) = \left(\prod_{j=1}^{N} \frac{v_j}{u_j}\right)^M \sum_{\{\lambda\}\subseteq(M)^N} S_{\{\lambda\}}(u_1^2,\dots,u_N^2) S_{\{\lambda\}}(v_1^{-2},\dots,v_N^{-2}) \tag{39}$$

2.7. **Restricting the 2-Toda tau-function to obtain the scalar product.**

Proposition 2. *The scalar product of the phase model for general N and M is, up to an overall factor of $\left(\prod_{j=1}^{N} \frac{v_j}{u_j}\right)^M$, a restricted τ-function of the 2-Toda hierarchy with $A_{\{\lambda\}\{\mu\}} = \delta_{\{\lambda\}\{\mu\}}$ and $s = n-M = m+N$, where m and n are free parameters.*

Proof. Beginning with the unrestricted τ-function,

$$\tau(s = n-M = m+N, \vec{x}, \vec{y}) = \sum_{\{\lambda\}\subseteq(M)^N} \chi_{\{\lambda\}}(\vec{x})\chi_{\{\lambda\}}(-\vec{y})$$

and performing the following change of variables,

$$x_k \to \frac{1}{k} p_k\left(u_1^2,\dots,u_N^2\right) \ , \ -y_k \to \frac{1}{k} p_k\left(v_1^{-2},\dots,v_N^{-2}\right) \ , 1 \le k \le N+M-1 \tag{40}$$

we obtain the required result. □

The above result only considers one value of s. Let us now consider the family of corresponding restricted τ-functions for other values of s. We begin by clarifying some known facts about the family of unrestricted τ-functions.

- The entire family is given by $\{\tau_{s=m+1}(\vec{x},\vec{y}),\tau_{s=m+2}(\vec{x},\vec{y}),\ldots,\tau_{s=n}(\vec{x},\vec{y})\}$.
- Different values of s **do not** change the amount of, (two sets of $n-m-1$), time variables.

We now compare this to the case of the family of restricted τ-functions.

- The initial τ-function, $\tau\left(s=n-M=m+N,\left\{u^2\right\},\left\{v^{-2}\right\}\right)$, has two sets of $N+M-1$ time variables, but each set is constructed from N symmetric variables.
- The introduction of the condition $s=n-M=m+N$ means that as s changes, so to do M and N. By considering the change in the dimensions of the partition, we can obtain how M and N change with s.

$$s\to s\pm l\Longleftrightarrow\left\{\begin{array}{l}M\to M\mp l\\ N\to N\pm l\end{array}\right. \tag{41}$$

- Consequently, although the number of time variables do not change with each s value, different values of s **do** change the amount of symmetric variables that the time variables are constructed from.

An example. Consider the complete family of unrestricted τ-functions for $n=5$ and $m=1$. In this case each τ-function contains two sets of 3 time variables, $\{\vec{x},\vec{y}\}=\{x_1,x_2,x_3,y_1,y_2,y_3\}$,

$$\left\{\tau_{s=2}=\sum_{\{\lambda\}\subseteq\{3\}}\Theta_{\{\lambda\}}\ ,\ \tau_{s=3}=\sum_{\{\lambda\}\subseteq\{2,2\}}\Theta_{\{\lambda\}}\ ,\ \tau_{s=4}=\sum_{\{\lambda\}\subseteq\{1,1,1\}}\Theta_{\{\lambda\}}\ ,\ \tau_{s=5}=\Theta_{\{\phi\}}\right\}$$

where $\Theta_{\{\lambda\}}=\chi_{\{\lambda\}}(\vec{x})\chi_{\{\lambda\}}(-\vec{y})$. The main question now is, if one τ-function in a family has been restricted to form a scalar product with a certain M and N value, *can the remaining τ-functions of the family also be restricted to form scalar products with valid M and N values?*

Naively performing the corresponding restrictions to the above family of τ-functions we obtain the following family of scalar products,

$$\left\{\gamma_1^3\mathbb{S}\begin{pmatrix}N=1\\ M=3\end{pmatrix},\gamma_2^2\mathbb{S}\begin{pmatrix}N=2\\ M=2\end{pmatrix},\gamma_3^1\mathbb{S}\begin{pmatrix}N=3\\ M=1\end{pmatrix},\gamma_4^0\mathbb{S}\begin{pmatrix}N=4\\ M=0\end{pmatrix}\right\}$$

where $\gamma_N^M=\left(\prod_{j=1}^N\frac{u_j}{v_j}\right)^M$. This example illustrates an important issue. We remember that each τ-function contained within a family must contain the same amount of time variables. Furthermore, it is a requirement that these time variables be the same for each value of s to ensure that the τ-functions obey the bilinear relation[9]. If this is to be the case for the above example, we have the following set of equations that must be satisfied,

$$p_k\left((u_1^i)^2\right)=p_k\left((u_1^{ii})^2,(u_2^{ii})^2\right)=\cdots=p_k\left((u_1^{iv})^2,\ldots,(u_4^{iv})^2\right)$$
$$p_k\left((v_1^i)^{-2}\right)=p_k\left((v_1^{ii})^{-2},(v_2^{ii})^{-2}\right)=\cdots=p_k\left((v_1^{iv})^{-2},\ldots,(v_4^{iv})^{-2}\right)$$

for $1\le k\le 3$. A simple check will reveal that only the trivial solution exists, meaning that all but one of the symmetric variables are set to zero. Thus, at a first glance, the answer to the question is no, due to the fact that *the τ-functions in the family all need to contain the same time variables, be they independent or restricted.*

We now generalize the above example.

[9]The τ-functions obviously **must** obey the bilinear relation.

Proposition 3. *The system of equations,* $0 \le l \le M-1$,

$$u_1^2 + \cdots + u_{N+l}^2 = \mu_1^2 + \cdots + \mu_N^2$$
$$u_1^4 + \cdots + u_{N+l}^4 = \mu_1^4 + \cdots + \mu_N^4$$
$$\vdots$$
$$u_1^{2(N+M-1)} + \cdots + u_{N+l}^{2(N+M-1)} = \mu_1^{2(N+M-1)} + \cdots + \mu_N^{2(N+M-1)}$$

permits only the trivial solution, i.e. $u_{\sigma_j}^2 = \mu_j^2$*, for* $j \in \{1, \ldots, N\}$*, and the remaining* l *of the* u_k^2*'s are equal to zero.*

Proof. We begin by considering the first $N+l$ equations in the system, the remaining equations will follow easily. We note that the left hand side of these polynomial equations exist in the symmetric polynomial ring $\mathbb{C}[u_1^2, \ldots, u_{N+l}^2]^{S_{N+l}}$. Consider now the polynomial ring $\mathbb{C}[s_1, \ldots, s_{N+l}]$, and recall that the fundamental theorem of symmetric polynomials states that there exists an isomorphism between the two rings, $\mathbb{C}[u_1^2, \ldots, u_{N+l}^2]^{S_{N+l}} \cong \mathbb{C}[s_1, \ldots, s_{N+l}]$, with the isomorphism sending[10] $p_j\left(u_1^2, \ldots, u_{N+l}^2\right) \to s_j$, $j = \{1, \ldots, N+l\}$. Hence the system of $N+l$ equations in the isomorphic polynomial ring, $\mathbb{C}[s_1, \ldots, s_{N+l}]$, is linear and has one solution.

Thus in the ring $\mathbb{C}[u_1^2, \ldots, u_{N+l}^2]^{S_{N+l}}$, the system contains one base solution, and every possible permutation of that base solution (since the polynomial ring is symmetric), leading to a total of $(N+l)!$ possible solutions. Since we already trivially know $(N+l)!$ solutions to the system, $u_{\sigma_j}^2 = \mu_j^2$ for $j \in \{1, \ldots, N\}$, and $u_{\sigma_k}^2 = 0$ for $k \in \{N+1, \ldots, N+l\}$, this means only the trivial solution exists for the first $N+l$ equations. It remains to note that the remaining $M-l-1$ equations are solved by the $(N+l)!$ solutions. □

Using the above result the following lemma comes almost automatically.

Lemma 1. *Assume we have a particular family of unrestricted* τ*-functions with particular* m *and* n *values,*

$$\{\tau_{m+1}(\vec{x},\vec{y}), \tau_{m+2}(\vec{x},\vec{y}), \ldots, \tau_n(\vec{x},\vec{y})\} \tag{42}$$

The process of restricting the entire family so that each τ*-function corresponds to a valid scalar product expression,*

$$\{\gamma_1^{n-m-1}\mathbb{S}\left(\begin{matrix} N=1 \\ M=n-m-1 \end{matrix}\middle| \vec{u},\vec{v}\right), \ldots, \gamma_{n-m}^{0}\mathbb{S}\left(\begin{matrix} N=n-m \\ M=0 \end{matrix}\middle| \vec{\mu},\vec{\nu}\right)\} \tag{43}$$

has potentially two (ill) effects.

- *If each of the above scalar product expressions has two sets of* N *(*N *is not constant for each scalar product) symmetric variables, then the 2 sets of* $n-m-1 = N+M-1$ *time variables of the restricted* τ*-functions are no longer equal, and therefore the* ***bilinear identity is no longer valid****.*
- *If we enforce that the time variables be equal, then we only have two sets of* ***one symmetric variable*** *for each of the scalar product expressions.*

Arguably both scenarios are pointless, so it makes sense to use the results of proposition 2 and only consider restricting one τ*-function in any family.*

Proof. Applying proposition 2 on all the unrestricted τ-functions in eq. 42, we instantly arrive to the expression in eq. 43. Analyzing any two of the above scalar product expressions, (with two sets of N symmetric variables), the results of proposition 3 state that the symmetric power sums, and hence the time variables, cannot be equal. Thus the first point in this lemma becomes obvious. Furthermore,

[10] We could use any basis symmetric polynomial, e_j, h_j, p_j, for the mapping. See section I.2 of [8] for further details.

from proposition 3, the only way for the time variables to be equal is if we trivialize the power sums as indicated in point 2 of this lemma. □

3. The Toda wave-vectors

In this section we shall show that restricting the wave-functions of 2-Toda hierarchy give an alternative method to calculating certain classes of correlation functions, and thus have a natural combinatorial meaning. In order to proceed we shall first give necessary definitions of skew Schur polynomials[11].

Skew Schur polynomials. Given a set of variables $\vec{u} = (u_1, \ldots, u_N)$ and the partitions $\{\lambda\}$, $\{\mu\}$, such that $\{\lambda\} \supseteq \{\mu\}$, the skew Schur polynomial, $S_{\{\lambda\}/\{\mu\}}(\vec{u})$, is defined as,

$$\begin{aligned} S_{\{\lambda\}/\{\mu\}}(\vec{u}) &= \sum_{T_{+}^{\{\lambda-\mu\}}} u_1^{t_1} u_2^{t_2} \ldots u_N^{t_N} = \sum_{T_{-}^{\{\lambda-\mu\}}} u_1^{t_1} u_2^{t_2} \ldots u_N^{t_N} \\ &= \det[h_{\lambda_i-\mu_j+j-i}(u_1, \ldots, u_N)]_{1 \leq i,j \leq N} \end{aligned} \tag{44}$$

where the sum is given over all possible (ascending or descending) semi-standard skew tableaux of shape $\{\lambda - \mu\}$, and the t_j give the weights of the tableau.

3.1. Wave-functions - I.

Considering the $\hat{w}^{(0)}$ class of wave-functions, from eq. 9 we obtain,

$$\tau(s)\hat{w}_k^{(0)}(s) = \sum_{\{\lambda\} \subseteq (n-(s+1))^{((s+1)-m)}} \chi_{\{\lambda\}}(\vec{x})\chi_{\{\lambda\}/\{k\}}(-\vec{y})$$

where we have used the following result,

$$\zeta_j(-\tilde{\partial}_{\vec{y}})\chi_{\{\lambda\}}(-\vec{y}) = \chi_{\{\lambda\}/\{j\}}(-\vec{y}) \tag{45}$$

for all partitions $\{\lambda\}$ such that $\{j\} \subseteq \{\lambda\}$.

Thus the upper triangular wave-matrix, $\hat{W}^{(0)}$, has entries of the form,

$$\hat{W}^{(0)} = \left(\frac{1}{\tau(j)} \sum_{\{\lambda\} \subseteq (n-(j+1))^{((j+1)-m)}} \chi_{\{\lambda\}}(\vec{x})\chi_{\{\lambda\}/\{k-j\}}(-\vec{y}) \right)_{m \leq j,k \leq n-1}$$

Constructing skew N-particle conjugate state vectors. Consider the following conjugate state vector,

$$\langle 0|\phi_k C(v_2) \ldots C(v_N) = \langle \Psi_M^{\{k\}}|$$

The allowable partitions of this conjugate vector are given by the following result,

Proposition 4.

$$\langle \Psi_M^{\{k\}}| = \sum_{\substack{\{\lambda\} \subseteq \{(M)^{(N-1)}, k\} \\ \{\lambda\} \supseteq \{k\}}} \psi_{\{\lambda\}}^{(1,k)} \langle \lambda|$$

Proof. Consider the non crossing column strict lattice path interpretation of the state vectors. The operator ϕ_k assures us that the first path in the first column makes a directional change from up to right at row k. This has the effect that the occupation number sequence will contain at least one entry n_l, where $l \geq k$. Transforming the occupation number sequence to a partition $\{\lambda\}$, we instantly receive the result, $\{\lambda\} \supseteq \{k\}$.

The fact that the first path in the first column turns right at row k also means that the highest row that the Nth path can be when it crosses between column N and $N+1$ is k. Thus the highest partition obtainable from lattice paths under this restriction are $\{\lambda\} = \{(M)^{(N-1)}, k\}$. □

[11] For more details see section I.V of [8].

Combinatorial definitions of $\psi^{(\mathbf{1,k})}_{\{\lambda\}}$. Considering the **lattice path** interpretation we receive,

$$\psi^{(1,k)}_{\{\lambda\}} = \sum_{\substack{\text{allowable paths in}\\ (M+1)\times N \text{ lattice}^{\dagger}}} v_2^{t_2^d - t_2^a} \dots v_N^{t_N^d - t_N^a}$$

where the lattice paths are under the condition that the first path in the first column makes a directional change from up to right at row k.

Considering the **lower diagonal plane partition** interpretation we receive,

$$\psi^{(1,k)}_{\{\lambda\}} = \sum_{\substack{\text{lower diag. plane part.}\\ \text{in } N\times N\times M \text{ array}^{\dagger}}} v_2^{l_2^d - l_2^a} \dots v_N^{l_N^d - l_N^a}$$

where the lower diagonal plane partitions are under the condition that the entry $\pi_{N,1}$ is equal to k.

Finally, considering the **ascending Young tableaux** interpretation, notice that when we transform from the lower diagonal plane partition to the Young tableau, the fact that $\pi_{N,1} = k$ means that the weight t_1 equals k. Since the weight t_1 does appear, as v_1 is not present, we can simply consider the skew partition $\{\lambda - k\}$ to generate the tableaux, leading to,

$$\psi^{(1,k)}_{\{\lambda\}} = \sum_{T_+^{\{\lambda-k\}}} v_2^{-2t_2+M} \dots v_N^{-2t_N+M} = (v_2 \dots v_N)^M S_{\{\lambda\}/\{k\}}(v_2^{-2}, \dots, v_N^{-2}) \tag{46}$$

3.2. **Boundary correlation functions - I.** Consider then the following boundary correlation function,

$$\langle \Psi_M^{\{k\}} | \Psi_M \rangle = \left(\frac{\prod_{j=2}^N v_j}{\prod_{j=1}^N u_j} \right)^M \sum_{\substack{\{\lambda\}\subseteq\{(M)^{(N-1)},k\}\\ \{\lambda\}\supseteq\{k\}}} S_{\{\lambda\}}(\{u^2\}) S_{\{\lambda\}/\{k\}}(v_2^{-2}, \dots, v_N^{-2}) \tag{47}$$

which calculates all the weighted non crossing column strict lattice paths on an $(M+1)\times 2N$ lattice with the first path in the first column turning right at row k. Compare it to any of the wave-functions calculated earlier, and concentrate on the particular row, $s = n - M - 1 = m + N - 1$, of the wave-matrix $\hat{W}^{(0)}$. If we restrict the variables in the usual way (eq. 40), and take the $v_1 \to \infty$ limit we immediately obtain,

$$\begin{aligned} \lim_{v_1\to\infty} \tau(n-M-1)\hat{w}_k^{(0)}(n-M-1) &= \sum_{\substack{\{\lambda\}\subseteq\{(M)^{(N-1)},k\}\\ \{\lambda\}\supseteq\{k\}}} S_{\{\lambda\}}(\{u^2\}) S_{\{\lambda\}/\{k\}}(\{v^{-2}\}) \\ &= \left(\frac{\prod_{j=1}^N u_j}{\prod_{j=2}^N v_j} \right)^M \langle \Psi_M^{\{k\}} | \Psi_M \rangle \end{aligned}$$

for $0 \le k \le M$. Thus the wave-vector, given by the $s = n - M - 1 = m + N - 1$ row of the wave-matrix, in the $v_1 \to \infty$ limit gives exactly (up to a multiplicative factor) all the weighted non crossing column strict lattice paths on an $(M+1)\times 2N$ lattice with the first path in the first column turning right at row k, $0 \le k \le M$.

Single determinant form for the wave-functions. When introducing the scalar product, we gave a single determinant form given by eq. 38. From this expression, it is possible to obtain a single determinant form for the wave-functions considered above[12].

[12]The details below are given in section VI of [5] to obtain single determinant expressions of boundary 1-point correlation functions. We expand upon these results shortly.

To achieve this, we first examine the operator $C(v)$ briefly. More explicitly, we are interested in the parts of $C(v)$ that contain only ϕ_j operators,

$$C(v) = \sum_{j=0}^{M} v^{M-2j}\phi_j + \text{terms that contain operators } \phi_j^\dagger \tag{48}$$

Thus when $C(v)$ acts on the conjugate vacuum we obtain,

$$\langle 0|C(v) = v^M \sum_{j=0}^{M} v^{-2j}\langle 0|\phi_j \tag{49}$$

We use this to express the scalar product as the following weighted linear sum of correlation functions,

$$\langle \Psi_M|\Psi_M\rangle = v_1^M \sum_{j=0}^{M} v_1^{-2j}\langle \Psi_M^{\{j\}}|\Psi_M\rangle \tag{50}$$

Therefore, if we expand the single matrix form for the scalar product as a polynomial in v_1^2, the coefficients will reveal a single matrix form for the correlation functions/wave-functions.

Using the following symmetric polynomial identity,

$$h_p(\{v^2\}, v_j^2) - h_p(\{v^2\}, v_k^2) = \left(v_j^2 - v_k^2\right) h_{p-1}(\{v^2\}, v_j^2, v_k^2) \tag{51}$$

where $\{v_j^2, v_k^2\} \notin \{v^2\}$, we can apply the following row operations,

$$R_{j_k} - R_{N-k+1} \ , \ 1 \le j_k \le N-k \ , \ 1 \le k \le N-1$$

to completely eliminate v_1 from the Vandermonde expression in the scalar product. Additionally, applying the following polynomial expansion of $h_p(\{v^2\}, v_j^2)$,

$$h_p(\{v^2\}, v_j^2) = \sum_{q=0}^{p} \left(v_j^2\right)^q h_{p-q}(\{v^2\})$$

to all entries in the determinant which contain v_1^2, we receive,

$$\langle \Psi_M|\Psi_M\rangle = v_1^M \sum_{q=0}^{M} v_1^{-2q}\Omega_{\hat{v}_1} \det \begin{bmatrix} h_q(u_k^2, v_2^2, \dots, v_N^2) \\ h_{M+N-1}(u_k^2, v_j^2) \end{bmatrix}_{\substack{j=2,\dots,N \\ k=1,\dots,N}} \tag{52}$$

where,

$$\Omega_{\hat{v}_1} = \prod_{i_1=1}^{N} u_{i_1}^{-M} \prod_{i_2=2}^{N} v_{i_2}^{-M} \prod_{1\le j_1<k_1\le N} \frac{1}{u_{j_1}^2 - u_{k_1}^2} \prod_{2\le j_2<k_2\le N} \frac{1}{v_{j_2}^2 - v_{k_2}^2} \tag{53}$$

which gives a single determinant form for the (restricted) wave-functions, $\hat{w}^{(0)}$.

3.3. **Wave-functions - II.** Considering the $\hat{w}^{(\infty)}$ class of wave-functions, using the definitions given previously we have,

$$\tau(s)\hat{w}_k^{(\infty)}(s) = (-1)^k \sum_{\{\lambda\}\subseteq(n-s)^{(s-m)}} \chi_{\{\lambda\}/\{1^k\}}(\vec{x})\chi_{\{\lambda\}}(-\vec{y})$$

where we have used the following result,

$$\zeta_j(-\tilde{\partial}_{\vec{x}})\chi_{\{\lambda\}}(\vec{x}) = (-1)^j \chi_{\{\lambda\}/\{1^j\}}(\vec{x}) \tag{54}$$

for all partitions $\{\lambda\}$ such that $\{\lambda\} \supseteq \{1^j\}$.

Thus the lower triangular wave-matrix $\hat{W}^{(\infty)}$ has the form,

$$\hat{W}^{(\infty)} = \left(\frac{(-1)^{j-k}}{\tau(j)} \sum_{\{\lambda\}\subseteq(n-j)^{(j-m)}} \chi_{\{\lambda\}/\{1^{j-k}\}}(\vec{x})\chi_{\{\lambda\}}(-\vec{y}) \right)_{m\le j,k\le n-1}$$

Constructing N-particle state vectors. Consider the following state vector,

$$B(u_1)\dots B(u_{N-k})\left(\phi_1^\dagger\right)^k |0\rangle = |\Psi_M^{\{1^k\}}\rangle$$

The allowable partitions of this vector are given by the following result.

Proposition 5.

$$|\Psi_M^{\{1^k\}}\rangle = \sum_{\substack{\{\lambda\}\subseteq\{(M)^{(N-k)},1^k\}\\ \{\lambda\}\supseteq\{1^k\}}} \psi_{\{\lambda\}}^{(2,1^k)}|\lambda\rangle \tag{55}$$

Proof. Consider again the lattice path interpretation of the state vectors. The operator(s) $\left(\phi_1^\dagger\right)^k$ assure us that the last k paths, labelled j_q, $N-k+1\leq q\leq N$, make directional changes from right to up at row 1, column q. Thus the largest occupation number sequence can be,

$$\{n_0,n_1,\ldots,n_M\}=\{0,k,0,\ldots,0,N-k\}\Rightarrow\{\lambda\}\subseteq\{(M)^{(N-k)},1^k\}$$

Also, since columns $\{N-k+1,\ldots,N\}$ only contain one $\phi^\dagger$ operator each, this means that only columns $\{1,\ldots,N-k\}$ can contain paths in the zeroth row. Due to the paths being column strict the lowest occupation number sequence is,

$$\{n_0,n_1,\ldots,n_M\}=\{N-k,k,0,\ldots,0,0\}\Rightarrow\{\lambda\}\supseteq\{1^k\}\ \square$$

Combinatorial definitions of $\psi_{\{\lambda\}}^{(2,1^k)}$. Considering the **lattice path** interpretation we obtain,

$$\psi_{\{\lambda\}}^{(2,1^k)} = \sum_{\substack{\text{allowable paths in}\\ (M+1)\times N\text{ lattice}^\dagger}} u_1^{t_1^d-t_1^a}\ldots u_{N-k}^{t_{N-k}^d-t_{N-k}^a}$$

where the lattice paths are under the condition that the last k paths, labelled j_q, $N-k+1\leq q\leq N$, make directional changes from right to up at row 1, column q, and only columns $\{1,\ldots,N-k\}$ can contain paths in the zeroth row.

Considering the **upper diagonal plane partition** interpretation we obtain,

$$\psi_{\{\lambda\}}^{(2,1^k)} = \sum_{\substack{\text{upper diag. plane part.}\\ \text{in } N\times N\times M\text{ array}^\dagger}} u_1^{l_1^d-l_1^a}\ldots u_{N-k}^{l_{N-k}^d-l_{N-k}^a}$$

where the upper plane partitions are under the condition that the top-right most $k\times k$ entries are equal to one. This obviously places restrictions on the remaining entries, as per the conditions of a plane partition. For example, the remaining $(N-k)\times k$ bottom-right entries can only either be zero or one accordingly.

Finally, whenever we biject from the upper plane partitions to the **descending Young tableaux**, the weights $t_{N-k+1}=\cdots=t_N=1$, and their position in the tableau are exactly $\{t_N=T_{1,1}^{\{\lambda\}},t_{N-1}=T_{2,1}^{\{\lambda\}},\ldots,t_{N-k+1}=T_{k,1}^{\{\lambda\}}\}$. Since these weights do not enter the equation, due to $u_{N-k+1},\ldots,u_N$ not being present, we can consider the skew partition $\{\lambda-1^k\}$ to generate the tableaux[13]. Thus we obtain,

$$\psi_{\{\lambda\}}^{(2,1^k)} = \sum_{T_-^{\{\lambda-1^k\}}} u_1^{2t_1-M}\ldots u_{N-k}^{2t_{N-k}-M} = \left(\frac{1}{u_1\ldots u_{N-k}}\right)^M S_{\{\lambda\}/\{1^k\}}(u_1^2,\ldots,u_{N-k}^2) \tag{56}$$

3.4. **Boundary correlation functions - II.** Consider then the boundary correlation function,

$$\langle\Psi_M|\Psi_M^{\{1^k\}}\rangle = \left(\frac{\prod_{j=1}^N v_j}{\prod_{j=1}^{N-k}u_j}\right)^M \sum_{\substack{\{\lambda\}\subseteq\{(M)^{(N-k)},1^k\}\\ \{\lambda\}\supseteq\{1^k\}}} S_{\{\lambda\}/\{1^k\}}(u_1^2,\ldots,u_{N-k}^2)S_{\{\lambda\}}(\{v^{-2}\}) \tag{57}$$

[13] Incidentally, it is at this point the reason we considered the tableaux in descending order becomes apparent. Had we considered ascending order we would need to invert the numbers to obtain the required results.

which calculates all the weighted non crossing column strict lattice paths on an $(M+1)\times 2N$ lattice with the final k paths, labelled j_q, $N-k+1\le q\le N$, turning up at row 1, column $N-k+1\le q\le N$. Additionally, only columns $1\le q\le N-k$ can contain paths in the zeroth row. Compare the above result with the $s=n-M=m+N$ row of the wave-matrix $\hat{W}^{(\infty)}$, restricting the variables as usual, and taking the $u_{N-k+1}=\cdots=u_N=0$ limit,

(58)
$$\lim_{\substack{u_j\to 0\\ N-k+1\le j\le N}} \tau(n-M)\hat{w}_k^{(\infty)}(n-M) = (-1)^k \sum_{\substack{\{\lambda\}\subseteq\{M^{(N-k)},1^k\}\\ \{\lambda\}\supseteq\{1^k\}}} S_{\{\lambda\}/\{1^k\}}(\{u^2\})S_{\{\lambda\}}(\{v^{-2}\})$$
$$= (-1)^k\left(\frac{\prod_{j=1}^{N-k}u_j}{\prod_{j=1}^{N}v_j}\right)^M \langle\Psi_M|\Psi_M^{\{1^k\}}\rangle$$

for $0\le k\le M$. Thus the wave-vector, given by the $s=n-M=m+N$ row of the wave-matrix, in the $u_{N-k+1}=\cdots=u_N=0$ limit gives exactly (up to a multiplicative factor) all the weighted non crossing column strict lattice paths on an $(M+1)\times 2N$ lattice with the final k paths, $1<k\le N$, labelled j_q, $N-k+1\le q\le N$, turning up at row 1, column $N-k+1\le q\le N$ and only the first $N-k$ columns can contain paths in the zeroth row.

Single determinant form for the wave-functions. We begin by examining the operator $B(u)$, as we are interested in the parts of $B(u)$ that contain only $\phi_j^\dagger$ and ϕ_1 operators,

$$B(u) = u^{-M}\left\{\sum_{j=0}^{M}u^{2j}\phi_j^\dagger + \sum_{j=0}^{M-2}u^{2j+2}\phi_0^\dagger\phi_1\phi_{j+2}^\dagger\right\} \tag{59}$$
$$+ \text{ terms that contain operators } \phi_j\ ,\ j\in\{2,\ldots,M\}$$

1-point boundary functions. Following the corresponding workings from section 3.2, we can obtain the scalar product as the following weighted linear sum of 1-point boundary correlation functions,

$$\langle\Psi_M|\Psi_M\rangle = u_N^{-M}\sum_{j=0}^{M}u_N^{2j}\langle\Psi_M|\Psi_M^{\{j\}}\rangle \tag{60}$$

Explicitly expanding the determinant expression as a polynomial in u_N^2 we obtain,

$$\langle\Psi_M|\Psi_M^{\{q\}}\rangle = \Omega_{\hat{u}_N}\det\left[h_{M+N-k}(\{u^2\}_k,v_j^2), h_{M-q}(\{u^2\}_{N-1},v_j^2)\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-1}} \tag{61}$$

where $\{u^2\}_k=\{u_1^2,\ldots,u_k^2\}$ and $\Omega_{\hat{u}_N}$ is the equivalent expression of eq. 53.

2-point boundary functions. We now build upon eq. 61 and use eq. 59 to consider the following quantity,

$$B(u_{N-1})\phi_1^\dagger|0\rangle = u_{N-1}^{-M}\sum_{j=0}^{M}u_{N-1}^{2j}\phi_j^\dagger\phi_1^\dagger|0\rangle + u_{N-1}^{-(M+2)}\sum_{j=2}^{M-2}u_{N-1}^{2j}\phi_0^\dagger\phi_j^\dagger|0\rangle \tag{62}$$

Hence we can express the 1-point correlation function $\langle\Psi_M|\Psi_M^{\{1\}}\rangle$, as the following linear sum of 2-point correlation functions,

$$\langle\Psi_M|\Psi_M^{\{1\}}\rangle = u_{N-1}^{-M}\left\{\langle\Psi_M|\Psi_M^{\{1,0\}}\rangle + u_{N-1}^{2M}\langle\Psi_M|\Psi_M^{\{M,1\}}\rangle\right\} \tag{63}$$
$$+\ u_{N-1}^{-M}\sum_{j=1}^{M-1}u_{N-1}^{2j}\left\{\langle\Psi_M|\Psi_M^{\{j,1\}}\rangle + \langle\Psi_M|\Psi_M^{\{j+1,0\}}\rangle\right\}$$

where the coefficient of u_{N-1}^{-M+2} is $\langle\Psi_M|\Psi_M^{\{1^2\}}\rangle + \langle\Psi_M|\Psi_M^{\{2,0\}}\rangle$. Thus in order to obtain $\langle\Psi_M|\Psi_M^{\{1^2\}}\rangle$, we need to first find $\langle\Psi_M|\Psi_M^{\{2,0\}}\rangle$.

This can be achieved by expanding $\langle\Psi_M|\Psi_M^{\{0\}}\rangle$ as a series in u_{N-1}^2,

$$\langle\Psi_M|\Psi_M^{\{0\}}\rangle = u_{N-1}^{-M}\sum_{j=0}^{M} u_{N-1}^{2j}\langle\Psi_M|\Psi_M^{\{j,0\}}\rangle$$

Substituting $q=0$ into eq. 61 and expanding as a polynomial in u_{N-1}^2 we obtain,

(64)
$$\langle\Psi_M|\Psi_M^{\{q,0\}}\rangle = \Omega_{\hat{u}_N,\hat{u}_{N-1}}\det\left[c_{jk}, h_{M+1}(\{u^2\}_{N-2}, v_j^2), h_{M-q}(\{u^2\}_{N-2}, v_j^2)\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-2}}$$

where $c_{jk} = h_{M+N-k}(\{u^2\}_k, v_j^2)$.

With the above result, we now expand the determinant form of $\langle\Psi_M|\Psi_M^{\{1\}}\rangle$, (the $q=1$ case of eq. 61), as a polynomial in u_{N-1}^2,

$$\begin{aligned}\langle\Psi_M|\Psi_M^{\{1\}}\rangle = u_{N-1}^{-M}\sum_{q_1=2}^{M+1}\sum_{q_2=0}^{1} u_{N-1}^{2q_1+2q_2-4}\Omega_{\hat{u}_N,\hat{u}_{N-1}}\\ \times\det\left[c_{jk}, h_{M-q_2+1}(\{u^2\}_{N-2}, v_j^2), h_{M-q_1+1}(\{u^2\}_{N-2}, v_j^2)\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-2}}\end{aligned}$$

where the indices $(q_1,q_2) = (q,1)$, and $(q+1,0)$, $2\le q\le M$, give us the sum $\langle\Psi_M|\Psi_M^{\{q-1,1\}}\rangle + \langle\Psi_M|\Psi_M^{\{q,0\}}\rangle$. Since we already have the explicit form of $\langle\Psi_M|\Psi_M^{\{q,0\}}\rangle$, given in eq. 64, this leaves us with the required result,

(65) $$\langle\Psi_M|\Psi_M^{\{q,1\}}\rangle = \Omega_{\hat{u}_N,\hat{u}_{N-1}}\det\left[c_{jk}, h_M(\{u^2\}_{N-2}, v_j^2), h_{M-q}(\{u^2\}_{N-2}, v_j^2)\right]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-2}}$$

n-point boundary functions. Given the previous examples, we present the following result.

Proposition 6.

(66)
$$\begin{aligned}\langle\Psi_M|\Psi_M^{\{r_1,\ldots,r_n\}}\rangle = \Omega_{\hat{u}_N,\ldots,\hat{u}_{N+1-n}}\, det\big[c_{jk}, h_{M-r_n+n-1}(\{u^2\}_{N-n}, v_j^2),\\ h_{M-r_{n-1}+n-2}(\{u^2\}_{N-n}, v_j^2),\ldots, h_{M-r_1}(\{u^2\}_{N-n}, v_j^2)\big]_{\substack{j=1,\ldots,N\\ k=1,\ldots,N-n}}\end{aligned}$$

where,

$$\begin{aligned} r_1\in\{0,1,\ldots,M\}\ ,\ r_2\in\{0,1\}\ ,\ \ldots,\ r_n\in\{0,1\}\\ r_1\ge r_2\ge\cdots\ge r_n\ ,\ 1\le n\le N\end{aligned}$$

Comment. The above statement can be proven using induction. By assuming that eq. 66 is true, we obtain $\langle\Psi_M|\Psi_M^{\{1^{n-q},0^q\}}\rangle$, $0\le q\le n-1$, as the following weighted sum of $(n+1)$-point correlation functions,

(67)
$$\begin{aligned}\langle\Psi_M|\Psi_M^{\{1^{n-q},0^q\}}\rangle = u_{N-n}^{-M}\langle\Psi_M|\Psi_M^{\{1^{n-q},0^{q+1}\}}\rangle + u_{N-n}^{M}\langle\Psi_M|\Psi_M^{\{M,1^{n-q},0^q\}}\rangle + u_{N-n}^{-M}\\ \times\sum_{j=1}^{M-1} u_{N-n}^{2j}\left\{\langle\Psi_M|\Psi_M^{\{j,1^{n-q},0^q\}}\rangle + \langle\Psi_M|\Psi_M^{\{j+1,1^{n-q-1},0^{q+1}\}}\rangle\right\}\end{aligned}$$

Additionally for $\langle\Psi_M|\Psi_M^{\{0^n\}}\rangle$, $(q=n)$, we have,

(68) $$\langle\Psi_M|\Psi_M^{\{0^n\}}\rangle = u_{N-n}^{-M}\sum_{j=0}^{M} u_{N-n}^{2j}\langle\Psi_M|\Psi_M^{\{j,0^n\}}\rangle$$

In order to verify the proposed result we need to derive, using polynomial expansion method(s) on eq. 66, the explicit determinant forms for the following expressions (referred to as step **1**, **2** and **3**),

- **Step 1)** $\langle\Psi_M|\Psi_M^{\{j,0^n\}}\rangle$, the coefficient of u_{N-n}^{-M+2j} in eq. 68.
- **Step 2)** $\langle\Psi_M|\Psi_M^{\{1^{n-q},0^{q+1}\}}\rangle$ and $\langle\Psi_M|\Psi_M^{\{M,1^{n-q},0^q\}}\rangle$, the coefficients of u_{N-n}^{-M} and u_{N-n}^{M} in eq. 67.

- **Step 3)** $\langle \Psi_M | \Psi_M^{\{j,1^{n-q},0^q\}} \rangle$ and $\langle \Psi_M | \Psi_M^{\{j+1,1^{n-q-1},0^{q+1}\}} \rangle$, the coefficients of u_{N-n}^{-M+2j} in eq. 67.

It is necessary in the third step to determine which of the determinant expressions corresponds to which correlation function. This is achieved through considering values of q where one term is already known from a previous result. As an example, consider $q = n-1$, where we receive the sum $\langle \Psi_M | \Psi_M^{\{j,1,0^{n-1}\}} \rangle + \langle \Psi_M | \Psi_M^{\{j+1,0^n\}} \rangle$. In this case we have already obtained $\langle \Psi_M | \Psi_M^{\{j+1,0^n\}} \rangle$ from the first step. For $q = n-2$ we receive the sum $\langle \Psi_M | \Psi_M^{\{j,1^2,0^{n-2}\}} \rangle + \langle \Psi_M | \Psi_M^{\{j+1,1,0^{n-1}\}} \rangle$, where we know the value of $\langle \Psi_M | \Psi_M^{\{j+1,1,0^{n-1}\}} \rangle$, $1 \leq j \leq M-2$, from the previous $(q = n-1)$ calculation, and we know $j = M-1$ from the second step. Carefully following this argument for all values of q we complete the proof by induction.

Thus using the results of eq. 66, we obtain the single determinant form for the restricted wave-functions, $\hat{w}^{(\infty)}$.

4. Discussion

The main result of this work is the correspondence between the 2-Toda wave-functions and the boundary correlation functions of the phase model. The weighted sum of the wave-functions analyzed in this work can be thought of as the action of a single vertex operator [10] on the 2-Toda τ-function,

$$\frac{\Gamma_x(\lambda)\tau(s,\vec{x},\vec{y})}{\tau(s,\vec{x},\vec{y})} = \sum_{k=0}^{s-m} \lambda^k \hat{w}_k^{(\infty)}(s) \ , \ \frac{\Gamma_y(\lambda)\tau(s+1,\vec{x},\vec{y})}{\tau(s,\vec{x},\vec{y})} = \sum_{k=0}^{n-s-1} \lambda^k \hat{w}_k^{(0)}(s)$$

where $\Gamma_{x/y}(\lambda) = \exp\left\{ -\sum_{k=1}^{n-m-1} \frac{\lambda^k}{k} \partial_{x_k} / \partial_{y_k} \right\}$. It is a pertinent question as to whether there is a combinatorial interpretation of a τ-function that has been acted on by more than one vertex operator. Alternatively, it is perfectly natural to speculate whether *non boundary* correlation functions of the phase model have natural correspondences with fundamental objects of the 2-Toda hierarchy. It is the author's intentions to examine these and related questions in the future.

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